

THE EIGENVALUES OF THE LAPLACIAN ON DOMAINS WITH SMALL SLITS

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ABSTRACT. We introduce a small slit into a planar domain and study the resulting effect upon the eigenvalues of the Laplacian. In particular, we show that as the length of the slit tends to zero, each real-analytic eigenvalue branch tends to an eigenvalue of the original domain. By combining this with our earlier work [HlrJdg07], we obtain the following application: The generic multiply connected polygon has simple spectrum.

1. INTRODUCTION

In this paper we study the following singular perturbation problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set having Lipschitz boundary. Remove from Ω the horizontal slit, $\Sigma_t = \{(x, y) \mid y = y_0, |x - x_0| \leq t\}$, centered at the point $(x_0, y_0) \in \Omega$. Standard analytic perturbation theory shows that the eigenvalues of the Laplacian on the slit domain $\Omega \setminus \Sigma_t$ depend analytically on t for $t > 0$. (See Theorem 4.2.) The main result of our study is the following.

Theorem 1.1. *Let $t \mapsto E_t$ be a real-analytic eigenvalue branch of the Laplacian acting on $L^2(\Omega \setminus \Sigma_t)$ with either Dirichlet or Neumann boundary conditions. Then E_t converges to an eigenvalue of the Laplacian acting on $L^2(\Omega)$ as t tends to 0.*

We remark that the convergence of real-analytic eigenbranches is much subtler than the convergence of ordered eigenvalues. For example, as t tends to zero, the k^{th} eigenvalue of the Laplacian on the rectangle $[0, t] \times [0, 1/t]$ tends to zero, but infinitely many analytic eigenvalue branches limit to infinity. Note that our proof that each eigenbranch in Theorem 1.1 has a finite limit is more involved than the proof that ordered eigenvalues limit to an eigenvalue of Ω . (Compare §3 and §6.)

This work began as part of a study of the Laplace spectrum of degenerating translation surfaces. Indeed, the collapse of a slit is one of the typical degenerations of a translation surface [EMZ03]. Theorem 1.1 deals with the simplest case of such a degeneration. The slit degeneration of a translation surface is analogous to the well-studied ‘neck pinching’ degeneration of hyperbolic surfaces and our overall strategy for proving Theorem 1.1 mirrors the strategy employed in [Wlp92] and [Jdg02].

The strategy is to control the negative variation of the logarithmic derivative, $\partial_t \log(E_t)$, for small t . A well-known formula—Proposition 4.6—relates $\partial_t E_t$ to certain quadratic forms defined as integrals over the domain. Not unexpectedly, one need only consider the contribution to these integrals that comes from a small neighborhood of the slit (Corollary 4.11). A judicious choice of coordinates in a neighborhood of the slit allows one to reduce the analysis of the quadratic forms to

a family of 1-dimensional problems indexed by $i = 0, 1, 2, 3 \dots$. A simple convexity estimate (Lemma 5.3) provides for control of the 1-dimensional contributions for $i > (2t)^2 \cdot E_t$ where $2t$ is the width of the slit. We first show that $t^2 \cdot E_t$ is bounded and then use convexity for i large and compactness for i small to find that $t^{2k} \cdot E_t$ tends to zero for some $k < 1$. Thus convexity applies to all $i > 0$, and a special estimate can be made for the case $i = 0$.

Elliptical coordinates in a neighborhood of the slit are particularly well-suited for our purpose. Define (r, θ) implicitly by

$$\begin{aligned} x - x_0 &= \sqrt{r^2 + t^2} \cdot \cos(\theta) \\ y - y_0 &= r \cdot \sin(\theta) \end{aligned}$$

The slit then corresponds exactly to the locus $r = 0$ and the level sets $r = \text{const}$ correspond to ellipses that surround the slit. If, in turn, one sets $r = t \sinh(z)$ then the equation $\Delta\psi = E \cdot \psi$ is separable in the variables (z, θ) . The solutions to the resulting ordinary differential equations are called Mathieu functions. Appendix A contains the basic facts about these functions that we use here. We note that the work of Y. Colin de Verdiere [CdV87] called our attention to the usefulness of these coordinates when considering slits, a fact that mathematical physicists have long been aware of [MrsRbn38] [MrsFsh].

We use Theorem 1.1 to extend the generic simplicity results of [HlrJdg07]. For example, we consider multiply connected polygons with n exterior vertices.¹ In §7, we prove the following:

Theorem 1.2. *If $n \geq 4$, then almost every multiply connected polygon has simple Laplace spectrum.*

In Theorem 1.2 we assume that the boundary conditions are either Neumann on every edge or Dirichlet on every edge. We also consider ‘mixed’ boundary conditions in the sense that the condition on each boundary segment is either Dirichlet or Neumann.

Theorem 1.3. *For each combinatorial choice of mixed Dirichlet-Neumann boundary conditions, almost every simply connected polygon with $n \geq 4$ vertices has simple Laplace spectrum.*

This latter theorem may be extended to multiply connected polygons with some mixed boundary conditions. It should be noted, however, that our method doesn’t give generic simplicity for every type of mixed boundary conditions in the case of multiply connected polygons. For instance, we cannot handle a slit with Dirichlet conditions on one side and Neumann conditions on the other.

We now outline the contents of this paper. In §2 we precisely describe the eigenvalue problem on a slit domain and give a fuller description of elliptical coordinates. In §3 we prove that sequences of eigenfunctions with uniformly bounded eigenvalues have convergent subsequences as the slit parameter t tends to zero. In §4 we apply Kato’s theory of analytic perturbations to show that the eigenvalues and eigenfunctions depend analytically on t . We also derive variational formula for these eigenvalues and prove that uniform estimates on the logarithmic derivative can be

¹One can regard such a polygon as the result of removing finitely many simply connected polygons from a simply connected n -gon.

‘localized’ to a neighborhood of the slit. In §5 we prove a convexity result for ‘radial’ Mathieu functions. In §6, we prove Theorem 1.1 according to the strategy describe above. In §7, we precisely state and then prove results concerning the spectral simplicity of polygons including Theorems 1.2 and 1.3.

2. THE EIGENVALUE PROBLEM ON A SLIT DOMAIN

Let $\Omega \subset \mathbb{R}^2$ be a compact domain with Lipschitz boundary, and let Σ be a disjoint union of line segments contained in the interior of Ω . The complement $\Omega \setminus \Sigma$ is not a domain with Lipschitz boundary, and hence one must take care in defining the eigenvalue problem. The purpose of this section is to describe a resolution that will be useful to us. We refer the reader to [Gri] for a complementary discussion of the analysis of singular domains.

To define the eigenvalue problem, we will regard a slit domain as the compact manifold with Lipschitz boundary obtained by completing $\Omega \setminus \Sigma$ with respect to the length metric. To be more precise, we define the distance $d(x, y)$ between x and y to be the infimum of lengths of rectifiable paths $\gamma : [0, 1] \rightarrow \Omega \setminus \Sigma$ that join x to y . Note that $d(x, y)$ equals the Euclidean distance between x and y iff x and y belong to a convex subset of $\Omega \setminus \Sigma$.

Definition 2.1. The *slit domain* Ω_Σ is the metric completion of $\Omega \setminus \Sigma$ with respect to d . We will refer to $\Omega_\Sigma \setminus (\Omega \setminus \Sigma)$ as the *slit*.

Although Ω_Σ is not isometric to a subdomain of \mathbb{R}^2 , it is naturally a compact Riemannian manifold with Lipschitz boundary. To see this, one can use elliptical coordinates to define a chart in a neighborhood of the slit.

2.1. Elliptical coordinates in a neighborhood of the slit. Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, and for each $t \geq 0$, define $\phi_t : \mathbb{R}^+ \times S^1 \rightarrow \mathbb{R}^2$

$$\phi_t(r, \theta) = \left(\sqrt{r^2 + t^2} \cdot \cos(\theta), r \sin(\theta) \right).$$

This injective map sends each circle $\{r\} \times S^1$ onto an ellipse with foci at $(\pm t, 0)$. We have $\phi_t(\mathbb{R}^+ \times S^1) = \mathbb{R}^2 \setminus \Sigma_t$ where $\Sigma_t = [-t, t] \times \{0\}$. By regarding $\mathbb{R}^2 \setminus \Sigma_t$ as a subset of $\mathbb{R}_{\Sigma_t}^2$, the map ϕ_t extends to a smooth diffeomorphism from $[0, \infty) \times S^1$ onto $\mathbb{R}_{\Sigma_t}^2$. Abusing notation slightly, we will use ϕ_t to denote this extension.

In these elliptical coordinates (r, θ) about the slit Σ_t , the gradient operator takes the form

$$(1) \quad \nabla_t w = \frac{r^2 + t^2}{r^2 + t^2 \sin^2(\theta)} \cdot \frac{\partial w}{\partial r} \cdot \partial_r + \frac{1}{r^2 + t^2 \sin^2(\theta)} \cdot \frac{\partial w}{\partial \theta} \cdot \partial_\theta,$$

and Lebesgue measure is expressed as

$$(2) \quad dm_t = \frac{r^2 + t^2 \sin^2(\theta)}{(r^2 + t^2)^{\frac{1}{2}}} dr d\theta.$$

The eigenvalue problem is unchanged if we rotate and/or translate the domain Ω and the slit $\Sigma \subset \Omega$ simultaneously. Thus, in the sequel, we will often make the following assumption.

Assumption 2.2. We assume that $\Sigma = [-t, t] \times \{0\}$.

Now choose $r_0 > 0$ so that the ball $B(\vec{0}, 2r_0)$ is contained in Ω . Then for $t < r_0$, the restriction of ϕ_t^{-1} to $[0, 2r_0) \times S^1$ provides a chart in a neighborhood of the slit.

2.2. The eigenvalue problem. Let dm denote Lebesgue measure. For each smooth function on the manifold Ω_Σ define

$$N(u) = \int_{\Omega} |u|^2 dm$$

and

$$q(u) = \int_{\Omega} |\nabla u|^2 dm.$$

Given a measurable subset, D , of the boundary of Ω_Σ , define $H_D^1(\Omega_\Sigma)$ to be the completion of

$$\{u \in C^\infty(\Omega_\Sigma) \mid u(m) = 0, \forall m \in D\}$$

with respect to the norm

$$u \mapsto q^{\frac{1}{2}}(u) + N^{\frac{1}{2}}(u).$$

The form q extends to a closed quadratic form on $H_D^1(\Omega_\Sigma)$ and the form N extends to a closed quadratic form on $L^2(\Omega_\Sigma, dm)$. In the sequel we will let $\mathbf{n}(\cdot, \cdot)$ (resp. $\mathbf{q}(\cdot, \cdot)$) denote the polarization of N (resp. q).

A function $\psi \in H_D^1(\Omega_\Sigma)$ is an *eigenfunction with eigenvalue E* if and only if

$$\mathbf{q}(\psi, v) = E \cdot \mathbf{n}(\psi, v)$$

for all $v \in H_D^1(\Omega_\Sigma)$. Integration by parts and standard elliptic estimates give that ψ is smooth in Ω_Σ with $\Delta\psi = E \cdot \psi$ where Δ is the Laplacian acting on smooth functions on \mathbb{R}^2 . Moreover, we have

- (1) $\psi(m) = 0$ for all $m \in \text{Int}(D)$ (Dirichlet conditions), and
- (2) $\nu(\psi)(m) = 0$ for all $m \in \text{Int}(\partial\Omega_\Sigma \setminus D)$ (Neumann conditions).

Here ν denotes the outward normal derivative along the boundary, $\partial\Omega_\Sigma$, of Ω_Σ . Note that in the sequel D will essentially be a union of segments.

3. CONVERGENCE OF EIGENFUNCTIONS WITH BOUNDED EIGENVALUES

Let Ω be a domain that contains the origin $\{0\}$, and let $t_n > 0$ be a sequence with $\lim_{n \rightarrow \infty} t_n = 0$. For sufficiently large n , the segment $\Sigma_{t_n} = [-t_n, t_n] \times \{0\}$ lies in the interior of Ω . For each $n \in \mathbb{N}$, let D_n be a measurable subset of the boundary of the slit domain $\Omega_{\Sigma_{t_n}}$ such that $D = D_n \cap \partial\Omega$ does not depend on n . Let ψ_n be a normalized eigenfunction of q on $H_{D_n}^1(\Omega_{\Sigma_{t_n}})$ with eigenvalue E_n . In this section we prove the following:

Theorem 3.1. *If E_n is a bounded sequence, then a subsequence $\psi_{n(k)}$ converges in $L^2(\Omega)$ to an eigenfunction ψ of q on $H_D^1(\Omega)$ with eigenvalue $E = \lim_{k \rightarrow \infty} E_{n(k)}$. Moreover, for every neighborhood U of the slit and every $j \in \mathbb{N}$, the sequence $\psi_{n(k)}$ converges to ψ in $C^j(\Omega \setminus U)$.*

Proof. Let U be a neighborhood of the slit. Each ψ_n is an eigenfunction of Δ with eigenvalue E_n , and hence for each n such that $\Sigma_{t_n} \subset U$ we have

$$\int_{\Omega \setminus U} |\psi_n \cdot \Delta^j \psi_n| dm = E_n^j \int_{\Omega \setminus U} |\psi_n|^2 dm.$$

Thus, using Gårding's inequality and the fact that ψ_n is normalized, we find that

$$\|\psi_n\|_j \leq C \cdot (E_n^k + 1)$$

where $\|\cdot\|_j$ is the norm associated to the Sobolev space $H^j(\Omega \setminus U)$. Since by assumption, E_n is a bounded sequence, we have that for each j , the sequence $n \mapsto \|\psi_n\|_j$ is bounded.

Using a diagonalization argument, we find a function ψ and a subsequence of ψ_n —still denoted ψ_n —such that for every j and every neighborhood U of the slit, ψ_n converges to ψ in the $H^j(\Omega \setminus U)$ norm. Thus, by the Sobolev embedding theorem, ψ_n converges to ψ in $C^k(\Omega \setminus U)$ for every $k \in \mathbb{N}$.

By assumption, the ψ_n have L^2 -norm equal to 1. Thus, given $\epsilon > 0$, Lemma 3.2 below applies to give $N_1 > 0$ and r^* so that if $n > N_1$, then

$$\int_B |\psi_n|^2 dm \leq \frac{\epsilon}{6}.$$

where B is some fixed ball centered at 0 and included in the neighborhood U_{t_n, r^*} for $n > N_1$. Hence, by Fatou's Lemma, we also have

$$\int_B |\psi|^2 dm \leq \frac{\epsilon}{6}.$$

The functions ψ_n converge uniformly to ψ on the complement of B , and hence there exists N_2 so that if $n > N_2$ then

$$\int_{\Omega \setminus U} |\psi - \psi_n|^2 dm < \frac{\epsilon}{3}.$$

It follows that ψ_n converges to ψ in $L^2(\Omega)$. In particular, ψ is L^2 -normalized and hence is nontrivial.

Finally, we show that ψ is an eigenfunction. To do this we adapt an argument from [CdV82]. Define the distribution T by $T(\phi) = \int_{\Omega} \psi \cdot \phi dm$. Then $\Delta' T \in H^{-2}(\Omega)$ where Δ' denotes the distributional Laplacian. If the support of ϕ does not contain the origin, then $\Delta' T(\phi) = E \cdot T(\phi)$, and hence the singular support of $\Delta' T - ET$ is contained in $\{0\}$. Hence there exists $S \in H^{-2}(\Omega)$ with $\text{supp}(S) \subset \{0\}$ such that

$$\Delta' T = E \cdot T + S.$$

It follows that there exists C such that $S = C \cdot \delta$ where $\delta(\phi) = \phi(0)$.

Let G be the distribution defined by $G(\phi) = \int_{\Omega} L \cdot \phi dm$ where $L(x, y) = \ln(\sqrt{x^2 + y^2})\chi(x, y)$ where χ is some smooth cut-off function near 0 (say for instance in the ball B). A direct computation shows that $\Delta' G - \delta$ is in $L^2(\Omega)$ and hence it follows from above that $\Delta'(T - C \cdot G)$ is also in $L^2(\Omega)$ and hence, by elliptic regularity, $T - C \cdot G \in H^2(\Omega)$.

It suffices to show that $C = 0$. On the one hand, by Fatou's Lemma, we have

$$\int_{\Omega} |\nabla \psi|^2 dm \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla \psi_n|^2 dm = E,$$

and on the other hand

$$\int_{\Omega} |\nabla L|^2 dm \geq 2\pi \int_0^\epsilon (\partial_r \ln(r))^2 r dr = \infty$$

where $B(0, \epsilon) \subset \Omega$ is a ball centered at the origin. Therefore, $T - C \cdot G \notin H^1(\Omega)$ unless $C = 0$. \square

Lemma 3.2. *Given $\epsilon > 0$ and $E_0 > 0$, there exists $r^* > 0$ such that if u is an eigenfunction of q on $H_D^1(\Omega_{\Sigma_t})$ with eigenvalue $E \leq E_0$, then for all $t < r^*$ we have*

$$\int_{U_{t,r^*}} |u|^2 dm \leq \epsilon \int_{\Omega} |u|^2 dm$$

where U_{t,r^*} is the elliptical neighborhood of the slit Σ_t of radius r^* .

Proof. For positive r and ρ , we have

$$\begin{aligned} |u(r + \rho, \theta) - u(r, \theta)| &\leq \int_0^\rho |\partial_r u(r + s, \theta)| ds \\ &\leq \sqrt{\rho} \cdot \left(\int_0^\rho |\partial_r u(r + s, \theta)|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. From this we find that

$$(3) \quad \frac{1}{2} \cdot |u(r, \theta)|^2 \leq |u(r + \rho, \theta)|^2 + \rho \int_0^\rho |\partial_r u(r + s, \theta)|^2 ds.$$

Given $r_1 > 0$, define $U_s = \{r \mid s \leq r \leq r_1 + s\} \times S^1$. Using (1) and (2), we find that if $s \geq 0$

$$\begin{aligned} \int_{U_0} |\partial_r u(r + s, \theta)|^2 dm &\leq \int_{U_0} |\partial_r u(r + s, \theta)|^2 (r^2 + t^2)^{\frac{1}{2}} d\theta dr \\ &\leq \int_{U_0} |\partial_r u(r + s, \theta)|^2 ((r + s)^2 + t^2)^{\frac{1}{2}} d\theta dr \\ &= \int_{U_s} |\partial_r u(r, \theta)|^2 (r^2 + t^2)^{\frac{1}{2}} d\theta dr. \\ &\leq \int_{U_s} |\nabla u(r, \theta)|^2 dm. \end{aligned}$$

Thus, since u is an eigenfunction with eigenvalue E ,

$$(4) \quad \rho \int_0^\rho \int_{U_0} |\partial_r u(r + s, \theta)|^2 dm ds \leq \rho^2 E \int_{\Omega} |u|^2 dm.$$

Fix some ρ so that $\rho^2 E \leq \epsilon/4$, and choose $r^* \leq \rho$. Uniformly for $r, t \leq r^*$, we have then

$$\frac{r^2 + t^2 \sin^2 \theta}{(r^2 + t^2)^{\frac{1}{2}}} \leq (r^2 + t^2)^{\frac{1}{2}} \leq r^*$$

and

$$\frac{(r + \rho)^2 + t^2 \sin^2 \theta}{(r + \rho)^2 + t^2)^{\frac{1}{2}}} \geq \frac{(r + \rho)^2}{(2(r + \rho)^2)^{\frac{1}{2}}} \geq \frac{\rho}{\sqrt{2}}$$

from which it follows that

$$\frac{r^2 + t^2 \sin^2 \theta}{(r^2 + t^2)^{\frac{1}{2}}} \leq \frac{r^* \sqrt{2}}{\rho} \cdot \frac{(r + \rho)^2 + t^2 \sin^2 \theta}{((r + \rho)^2 + t^2)^{\frac{1}{2}}}.$$

We now fix $r^* \leq \frac{\epsilon \rho}{\sqrt{32}}$ so that we have

$$\frac{r^2 + t^2 \sin^2 \theta}{(r^2 + t^2)^{\frac{1}{2}}} \leq \frac{\epsilon}{4} \cdot \frac{(r + \rho)^2 + t^2 \sin^2 \theta}{((r + \rho)^2 + t^2)^{\frac{1}{2}}}$$

uniformly for $r, t \leq r^*$.

Thus, using (1) and (2), we have

$$\begin{aligned}
\int_{U_0} |u(r + \rho, \theta)|^2 dm &\leq \frac{\epsilon}{4} \int_{U_0} |u(r + \rho, \theta)|^2 \frac{(r + \rho)^2 + t^2 \sin^2 \theta}{((r + \rho)^2 + t^2)^{\frac{1}{2}}} d\theta dr \\
&= \frac{\epsilon}{4} \int_{U_\rho} |u(r, \theta)|^2 \frac{r^2 + t^2 \sin^2 \theta}{(r^2 + t^2)^{\frac{1}{2}}} d\theta dr \\
&\leq \frac{\epsilon}{4} \int_{\Omega} |u(r, \theta)|^2 dm.
\end{aligned}$$

By combining this with (3) and (4) we find that

$$\frac{1}{2} \int_{U_0} |u(r, \theta)|^2 dm \leq \left(\frac{\epsilon}{4} + \rho^2 E \right) \int_{\Omega} |u(r, \theta)|^2 dm.$$

The claim follows since $\rho^2 E \leq \epsilon/4$. \square

4. ANALYTICITY, VARIATIONAL FORMULAE, AND LOCALIZATION

In this section we use standard perturbation theory to show that the eigenvalues and eigenfunctions depend analytically on t for $t > 0$. We then derive basic formulae relating the derivative of an eigenbranch to the derivatives of the quadratic forms q and N . We show that for small t the dominant terms in the derivative formulae can be localized to a neighborhood of the slit. Finally, we evaluate these local formulae on an elliptical neighborhood of the slit.

4.1. Pulling back to elliptical coordinates. Standard analytic perturbation theory [Kato] applies to a family of quadratic forms on a *fixed* Hilbert space. For this reason we will modify the family quadratic forms q and N but in such a way that we obtain an equivalent eigenvalue problem.

Of the various possible approaches, we choose to modify the map ϕ_t so that the inverse image of Ω_{Σ_t} is constant in t . The fixed Hilbert space will then be L^2 on this inverse image with respect to the pull-back of Lebesgue measure.

Let r_0 be as in §2. Namely, $B(\vec{0}, 2r_0) \subset \Omega$. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth, positive, decreasing function such that $\sigma(t) = 1$ if $|t| \leq 1$ and $\sigma(t) = 0$ if $|t| \geq 2$. Define $\tilde{\phi}_t : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^2$ by

$$\tilde{\phi}_t(r, \theta) = \left(\sqrt{r^2 + t^2 \cdot \sigma(r/r_0)} \cdot \cos(\theta), r \sin(\theta) \right).$$

By construction, $\tilde{\phi}_t$ defines a smooth diffeomorphism from $[0, \infty) \times \mathbb{R}$ onto $\mathbb{R}_{\Sigma_t}^2$.

If $t < r_0$, then the set $M = \tilde{\phi}_t^{-1}(\Omega_t)$ does not depend on t . We pull-back the Dirichlet energy functional and the L^2 -norm on Ω to functionals on M .

For computational convenience, we express the change of variables in the language of Riemannian geometry. For each t , define the Riemannian metric

$$g_t = \tilde{\phi}_t^*(dx^2 + dy^2)$$

on $[0, \infty) \times S^1$. In particular, $\tilde{\phi}$ is a Riemannian isometry from $(\Omega_t, dx^2 + dy^2)$ onto (M, g_t) .

Let dm_t denote the Riemannian measure: $dm_t = (\tilde{\phi}_t)_*^{-1}(dxdy)$. The space, $L^2(M)$, of functions that are square integrable with respect to dm_t does not depend

on t . On the other hand, the associated norm does depend on t . Let N_t denote the quadratic form²

$$(5) \quad N_t(u) = \int_M |u|^2 dm_t.$$

In the sequel we will let $\mathbf{n}_t(\cdot, \cdot)$ denote the polarization of N_t .

Let ∇_t denote the Riemannian gradient for (M, g_t) . Namely, we have $g_t(\nabla_t u, X) = du(X)$ for every smooth function $u : M \rightarrow \mathbb{R}$ and vector field X on M . For smooth functions $u : M \rightarrow \mathbb{R}$ define

$$(6) \quad q_t(u) = \int_M g_t(\nabla_t u, \nabla_t u) dm_t.$$

Given a measurable subset D in the boundary of M , define $H_D^1(M)$ to be the completion with respect to the norm

$$u \mapsto q_t(u)^{\frac{1}{2}} + N_t(u)^{\frac{1}{2}}.$$

of the set of smooth functions on M that vanish on D . Since the equivalence class of this norm is independent of $t > 0$, the completion does not depend on t . The form q_t extends to a closed quadratic form on $H_D^1(M)$. In the sequel we will let $\mathbf{q}_t(\cdot, \cdot)$ denote the polarization of q_t .

Proposition 4.1. *ψ is an eigenfunction of the quadratic form q on $H_{\phi_t(D)}^1(\Omega_{\Sigma_t})$ with respect to N on $L^2(\Omega_{\Sigma_t})$ with eigenvalue E if and only if $u = \tilde{\phi}_t^*(\psi)$ is a eigenfunction of q_t on $H_D^1(M)$ with respect to N_t on $L^2(M)$ with eigenvalue E .*

Proof. This follows from a straightforward accounting of the change of variables given by $\tilde{\phi}_t$. \square

4.2. Analyticity of eigenvalues and eigenfunctions.

Proposition 4.2. *The eigenvalues and eigenfunctions of q_t with respect to N_t vary real-analytically. To be precise: $\forall k \in \mathbb{N}$, \exists real-analytic paths $\psi_k : (0, t_0] \rightarrow H_D^1(M)$ and $E_k : (0, t_0] \rightarrow \mathbb{R}$ such that*

- (a) $\mathbf{q}_t(\psi_k(t), v) = E_k(t) \cdot \mathbf{n}_t(\psi_k(t), v)$ for all $v \in H_D^1(M)$.
- (b) $N_t(\psi_k(t)) = 1$, and
- (c) the span of $\{\psi_k(t) \mid k \in \mathbb{N}\}$ is dense in $L^2(M)$.

Proof. Apply standard analytic perturbation theory. In particular, both q_t and N_t are holomorphic families of quadratic forms of type (a) as in §VII.4 [Kato]. The eigenvalue problem is of generalized form as in §VII.6 [Kato]. (See especially Remark VII.6.2 [Kato] and the discussion on page 419.) The analogue of Remark VII.4.22 [Kato] gives the claim. \square

Remark 4.3. We will use the expression *eigenbranch* to designate the mapping $t \rightarrow (\psi_k(t), E_k(t))$ and will use the expression *normalized eigenbranch* if, in addition, $N_t(\psi_k(t)) = 1$. By extension, the term *eigenbranch* may also refer to either the eigenvalue or the eigenvector singly. Since we will be dealing with one eigenbranch at a time, the index k will be systematically dropped.

²This quadratic form is defined on functions on M , and hence is not the same as the quadratic form N defined in §2. On the other hand, these forms differ by pull-back by ϕ_t , and the context will make clear which is being used.

Corollary 4.4. *The Neumann (resp. Dirichlet) eigenvalues of the Laplacian on Ω_{Σ_t} depend analytically on $t \in (0, t_0]$.*

Proof. Combine Proposition 4.1, Proposition 4.2, and the discussion in §2. \square

4.3. Variational formulae.

Notation 4.5. A dot above a quantity will indicate differentiation with respect to t . For example, \dot{E} indicates the first derivative of an eigenvalue branch E_t . In what follows, we will often suppress the dependence of q , ψ , and E on t from the notation.

We begin with a well-known, general variational formula.

Proposition 4.6. *We have*

$$\dot{E} \cdot N(\psi) = \dot{q}(\psi) - E \cdot \dot{N}(\psi).$$

Proof. Substitution of $\dot{\psi}$ for v in part (a) of Proposition 4.2 gives

$$(7) \quad \mathbf{q}(\psi, \dot{\psi}) = E \cdot \mathbf{n}(\psi, \dot{\psi}).$$

By differentiating part (a) of Proposition 4.2, we obtain

$$\dot{\mathbf{q}}(\psi, v) + \mathbf{q}(\dot{\psi}, v) = \dot{E} \cdot \mathbf{n}(\psi, v) + E \cdot (\dot{\mathbf{n}}(\psi, v) + \mathbf{n}(\dot{\psi}, v))$$

for all v . By substituting ψ for v and using (7), we find that

$$\dot{q}(\psi, \psi) = \dot{E} \cdot N(\psi) + E \cdot \dot{N}(\psi).$$

\square

In the case that q is the Dirichlet energy associated to a family of Riemannian metrics, the quantities in Proposition 4.6 can be expressed in terms of the first variation of the metric and the associated Riemannian measure.

Proposition 4.7. *Let $t \rightarrow g_t$ be a real-analytic family of Riemannian metrics on M , and suppose that q is defined by (6). Then we have*

$$\dot{q}(u) = - \int_M \dot{g}(\nabla u, \nabla u) \, dm + \int_M g(\nabla u, \nabla u) \, \dot{dm}$$

and

$$\dot{N}(u) = \int_M |u|^2 \, \dot{dm}.$$

Proof. By differentiating (6) we have

$$\dot{q}(u) = \int_M \dot{g}(\nabla u, \nabla u) \, dm + 2 \int_M g(\nabla u, \dot{\nabla} u) \, dm + \int_M g(\nabla u, \nabla u) \, \dot{dm}.$$

By definition $g(\nabla u, X) = X(f)$ for all vector fields X , and hence we have

$$(8) \quad \dot{g}(\nabla u, X) + g(\dot{\nabla} u, X) = 0.$$

In particular, if $X = \nabla u$, then we have

$$\int_M \dot{g}(\nabla u, \nabla u) \, dm + \int_M g(\dot{\nabla} u, \nabla u) \, dm = 0.$$

Substitution into the formula for \dot{q} gives the first formula. The second formula follows from (5). \square

The following lemma will be used to translate estimates on the logarithmic derivative of an eigenbranch into statements concerning the convergence of the eigenbranch.

Lemma 4.8. *If there exists t_0 and a continuous positive function $\rho : (0, t_0] \rightarrow \mathbb{R}^+$ such that*

$$\partial_t E_t \geq -\rho(t) \cdot E_t$$

for all $t \in (0, t_0]$, then the function

$$F(t) = \exp\left(-\int_t^{t_0} \rho(s) ds\right) \cdot E_t$$

converges as t tends to zero. In particular, if ρ is integrable, then E_t converges as t tends to zero.

Proof. We have

$$F'(t) = (\partial_t E_t + \rho(t) \cdot E_t) \cdot \exp\left(-\int_t^{t_0} \rho(s) ds\right) \geq 0,$$

and hence F is increasing. Since F is nonnegative, the claim follows. \square

Using this lemma and Proposition 4.6 we have the following corollary.

Corollary 4.9. *Suppose that there exists a constant C such that for all $u \in H^1(M)$ we have*

$$(9) \quad \dot{q}(u) \geq -C \cdot q(u)$$

and

$$(10) \quad \dot{N}(u) \leq C \cdot N(u).$$

Then E_t converges as t tends to 0.

Proof. From Proposition 4.6 we have $\dot{E} \geq -2C \cdot E$. Thus we can apply Lemma 4.8 with $\rho \equiv 2C$. \square

4.4. A localization principle. Unfortunately, inequalities (9) and (10) do not hold true for the singular perturbation that we consider here. We now show, however, that these inequalities do hold for all u with support outside of a neighborhood of the slit (Proposition 4.10).

To state this result in a convenient form, we introduce the following notation. Let U be a measurable set.³ For $w \in H^1$, we denote by $q_U(w)$ the ‘restriction’ of q to U . That is,

$$q_U(w) = \int_U |\nabla w|^2 dm.$$

We will use analogous notation for the quadratic forms \dot{q} , N , and \dot{N} .

Proposition 4.10. *Let $U \subset \Omega$ be a neighborhood of the slit. There exists a constant C_U such that for any $w \in H^1$*

$$\dot{q}_{M \setminus U}(w) \geq -C_U \cdot q(w)$$

and

$$\dot{N}_{M \setminus U}(w) \leq C_U \cdot N(w).$$

³In the sequel, U will often be an elliptical neighborhood of the slit.

Proof. Let SM denote the unit tangent bundle to M (with respect to g_{t_0}), and for each small t , define $F_t : SM \rightarrow \mathbb{R}$ by

$$F_t(X) = \frac{\frac{d}{dt}g_t(X, X)}{g(X, X)}.$$

The restriction of g_t to $\Omega \setminus U$ is real-analytic for $t \in [0, t_0]$, and hence

$$C_1 = \sup\{F_t(X) \mid t \in [0, t_0] \text{ and } X \in S(\Omega \setminus U)\}$$

is finite. By homogeneity of g and \dot{g} in each tangent space, we have $\dot{g}(X, X) \leq C_1 \cdot g(X, X)$ for all $X \in T(\Omega \setminus U)$. Therefore,

$$\int_{M \setminus U} \dot{g}(\nabla w, \nabla w) \, dm \leq C_1 \int_{M \setminus U} g(\nabla w, \nabla w) \, dm$$

Let $G_t : [0, r_0] \times M \rightarrow \mathbb{R}$ be defined by

$$G_t(p) = \frac{\frac{d}{dt}dm_t}{dm_t}(p).$$

Since dm_t restricted to $\Omega \setminus U$ depends real-analytically on $t \in [0, t_0]$,

$$C_2 = \sup\{|f(t, X)| \mid t \in [0, r_0] \text{ and } X \in \Omega \setminus U\}$$

is finite. It follows that

$$\int_{M \setminus U} w^2 \, dm \leq C_2 \int_{M \setminus U} w^2 \, dm,$$

and

$$\int_{M \setminus U} g(\nabla w, \nabla w) \, dm \geq -C_2 \int_{M \setminus U} g(\nabla w, \nabla w) \, dm.$$

We use Proposition 4.7 and set $C_U = C_1 + C_2$. The result then follows. \square

Corollary 4.11. *Let U be a neighborhood of the slit. Suppose that there exist functions $\alpha, \beta, \gamma : [0, \infty) \rightarrow [0, \infty)$ such that the following holds for an eigenbranch ψ*

$$\dot{q}_U(\psi) \geq -\alpha(t) \cdot q_U(\psi) - \beta(t) \cdot E \cdot N_U(\psi),$$

and

$$\dot{N}_U(\psi) \leq \gamma(t) \cdot N_U(\psi).$$

Then there exists $C' > 0$ such that

$$\dot{E} \geq -(C' - \alpha(t) - \beta(t) - \gamma(t)) E.$$

Proof. We have

$$\begin{aligned} \dot{q}(\psi) &= \dot{q}_{M \setminus U}(\psi) + \dot{q}_U(\psi) \\ &\geq -C \cdot q(\psi) - \alpha \cdot q_U(\psi) - \beta \cdot E \cdot N_U(\psi) \\ &\geq -(C + \alpha) \cdot q(\psi) - \beta \cdot E \cdot N(\psi) \\ &\geq -(C + \alpha + \beta) \cdot E \cdot N(\psi) \end{aligned}$$

where the constant C comes from Proposition 4.10. Similarly, we find that

$$\dot{N}_U(\psi) \leq (C + \gamma) \cdot N_U(\psi).$$

The estimate then follows from Proposition 4.6 with $C' = 2C$. \square

Remark 4.12. Proposition 4.10 and Corollary 4.11 represent a *localization principle* in the sense that an estimate of the logarithmic derivative of E_t has been reduced to the study of the functionals \dot{q} and \dot{N} in a small neighborhood of the slit. This principle applies more generally to singular perturbation problems in which the ‘singular support’ of the perturbation is small.

4.5. Evaluation on the ellipse. We now let U be the elliptical neighborhood of the slit of radius r_0 and evaluate q_U and N_U . By using the expression for the gradient (1) and the Lebesgue measure (2) in elliptical coordinates, we find that

$$q_U(v) = \int_U |\partial_r v|^2 (r^2 + t^2)^{\frac{1}{2}} dr d\theta + \int_U |\partial_\theta v|^2 \frac{dr d\theta}{(r^2 + t^2)^{\frac{1}{2}}}$$

and

$$N_U(v) = \int_U |v|^2 \frac{(r^2 + t^2 \sin^2(\theta))}{(r^2 + t^2)^{\frac{1}{2}}} dr d\theta.$$

By Proposition 4.7 or direct computation, we find that

$$\dot{q}_U(v) = t \int_U |\partial_r v|^2 \frac{dr d\theta}{(r^2 + t^2)^{\frac{3}{2}}} - t \int_U |\partial_\theta v|^2 \frac{dr d\theta}{(r^2 + t^2)^{\frac{3}{2}}}$$

and

$$\dot{N}_U(v) = t \int_U |v|^2 \frac{((2 \sin^2(\theta) - 1)r^2 + t^2 \sin^2(\theta))}{(r^2 + t^2)^{\frac{3}{2}}} dr d\theta.$$

It will prove convenient to make a change of variables.

Notation 4.13. In the following, we let

- $r = t \cdot \sinh x$,
- $Y_t = \sinh^{-1}(r_0/t)$, and
- $U_t = [0, Y_t] \times S^1$.

With this change of coordinates, the formulae above become

$$(11) \quad q_U(v) = \int_{U_t} |\partial_x v|^2 dx d\theta + \int_{U_t} |\partial_\theta v|^2 dx d\theta,$$

$$(12) \quad N_U(v) = t^2 \int_{U_t} |v|^2 (\sinh^2 x + \sin^2(\theta)) dx d\theta,$$

$$(13) \quad \dot{q}_U(v) = \frac{1}{t} \int_{U_t} \frac{|\partial_x v|^2}{\cosh^2 x} dx d\theta - \frac{1}{t} \int_{U_t} \frac{|\partial_\theta v|^2}{\cosh^2 x} dx d\theta,$$

and

$$(14) \quad \dot{N}_U(v) = t \int_{U_t} |v|^2 (\sin^2(\theta) - \cosh^2(x) \tanh^2(x)) dx d\theta.$$

From (13) and (14) we find that

$$(15) \quad \dot{q}_U(v) \geq -\frac{1}{t} \int_{U_t} \frac{|\partial_\theta v|^2}{\cosh^2 x} dx d\theta,$$

and

$$(16) \quad \dot{N}_U(v) \leq t \int_{U_t} |v|^2 \sin^2(\theta) dx d\theta \leq t \int_{U_t} |v|^2 dx d\theta$$

5. MATHIEU FUNCTIONS AND INTEGRAL ESTIMATES

In this section we provide estimates that will be used to control ratios such as \dot{N}/N and \dot{q}/q in the next section.

5.1. Separation of variables. E. Mathieu [Mth68] observed that one can apply the method of separation of variables to the eigenvalue problem, $\Delta\psi = E \cdot \psi$, on an ellipse in \mathbb{R}^2 . He made a detailed study of the solutions to the resulting ordinary differential equations, solutions that now bear his name. To perform the separation of variables, we use the same change of variables as in the preceding section, i.e. we set

$$r = t \cdot \sinh(x).$$

Let $h \in \mathbb{R}$. The operator

$$-\frac{\partial^2}{\partial \theta^2} + h^2 \cos^2(\theta)$$

acting on $L^2(\mathbb{R}/(2\pi\mathbb{Z}), d\theta)$ has discrete spectrum

$$0 \leq b_0(h) \leq b_1(h) \leq b_2(h) \leq \dots$$

For $h \neq 0$, the spectrum is simple [MrsFsh]. Let $v_{i,h}$ denote the L^2 -normalized eigenfunction associated to $b_i(h)$. We will call $v_{i,h}$ the i^{th} *angular Mathieu function*. In the following we will often suppress the dependence of $v_{i,h}$ on h from the notation.

For the convenience of the reader, we prove the following in Appendix A.

Proposition 5.1. *Suppose that $\Delta\psi = E \cdot \psi$ on the ellipse of radius $r_0 = t \sinh(x_0)$, and suppose that ψ satisfies a Dirichlet (resp. Neumann) boundary condition on the slit: For $0 \leq \theta \leq 2\pi$ we have*

$$\psi(0, \theta) \equiv 0 \quad (\text{resp. } \partial_x \psi(0, \theta) \equiv 0).$$

Then for $x \geq 0$,

$$(17) \quad \psi(x, \theta) = \sum_i u_i(x) \cdot v_i(\theta)$$

where

$$h = t \cdot \sqrt{E},$$

and where $u_i : [0, x_0] \rightarrow \mathbb{R}$ is a solution to

$$(18) \quad -u''(x) + (b_i(h) - h^2 \cosh^2(x)) \cdot u(x) = 0$$

with $u(0) = 0$ (resp. $u'(0) = 0$).

A solution u to (18) will be called a *radial Mathieu function*.

In the sequel, our methods will rely upon estimates of the distribution of the L^2 and H^1 mass of an eigenfunction on the ellipse. The following lemma will allow us to reduce such estimates to estimates of radial Mathieu functions.

Lemma 5.2. *We have*

$$(19) \quad \int_{S^1} |\psi|^2 d\theta = \sum_{i=0}^{\infty} |u_i(x)|^2.$$

$$(20) \quad \int_{S^1} |\partial_\theta \psi|^2 d\theta + h^2 \int_{S^1} |\psi|^2 \cos^2(\theta) d\theta = \sum_{i=0}^{\infty} b_i(h) \cdot |u_i(x)|^2.$$

Proof. By standard Sturm-Liouville theory, the angular Mathieu functions $\{v_i\}$ form a complete and orthogonal set in $L^2(S^1, d\theta)$. (See Appendix A). Thus, (19) follows from (17).

From (17), we have $\partial_\theta \psi = \sum u_i \cdot \partial_\theta v_i$. Since v_i is an eigenfunction with eigenvalue $b_i = b_i(h)$ we have

$$-\partial_\theta^2 v_i + h^2 \cos^2(\theta) \cdot v_i = b_i \cdot v_i.$$

By multiplying by v_j and integrating by parts, we have

$$\int_{S^1} \partial_\theta v_i \cdot \partial_\theta v_j \, d\theta + h^2 \int_{S^1} v_i \cdot v_j \cos^2(\theta) \, d\theta = b_i \int_{S^1} v_i \cdot v_j \, d\theta.$$

Equation (20) then follows from the fact that $\{v_i\}$ is complete and orthogonal. \square

5.2. Radial convexity estimates. Our estimates of the distribution of the L^2 mass of a radial Mathieu function u depend upon the following observation. We have

$$(u^2)'' = 2 \cdot u \cdot u'' + 2 \cdot (u')^2 \geq 2 \cdot u \cdot u'',$$

and hence if $b_i - h^2 \cosh^2(x) \geq \frac{1}{2}$, then by (18) we have

$$(u^2)'' \geq u^2.$$

We will let $w : [0, X] \rightarrow \mathbb{R}^+$ denote a smooth function such that

$$w''(x) \geq w(x),$$

$$w(x) \geq 0$$

for all $x \in [0, X]$ and

$$w'(0) \geq 0.$$

In particular, if X satisfies

$$(21) \quad b_i - h^2 \cdot \cosh^2(X) \geq \frac{1}{2},$$

then the square, u_i^2 , of a radial Mathieu function satisfying either Dirichlet or Neumann conditions is an example of such a function w .

The following expression of convexity is the basis for our estimates.

Lemma 5.3. *For all $x, y \geq 0$ such that $x + y \leq X$, we have*

$$w(x + y) \geq w(x) \cdot \cosh(y).$$

Proof. The claim holds if $w(x) = 0$. So we may assume that $w(x) > 0$. Let $z(y) = w(x + y)/w(x)$. Note that $z(0) = 1$ and $z'(y) \geq 0$. Since $w''(x + y) \geq w(x + y)$, we have $\frac{d^2}{dy^2}(z(y) - \cosh(y)) \geq 0$ and since $w'(x) \geq 0$, we have $\frac{d}{dy} \Big|_{y=0} (z(y) - \cosh(y)) \geq 0$. It follows that $\partial_y(z(y) - \cosh(y)) \geq 0$ for all $y \geq 0$. Since $z(0) - \cosh(0) = 0$, we have $z(y) - \cosh(y) \geq 0$ for all y , and the result follows. \square

Proposition 5.4. *Let $p : [0, X] \rightarrow \mathbb{R}^+$ be a decreasing integrable function. Then*

$$\int_0^X p(x) \cdot w(x) \, dx \leq \left(\frac{p(0)}{\cosh(X/2)} + p(X/2) \right) \int_0^X w(x) \, dx.$$

Proof. Applying Lemma 5.3 with $y = X/2$ gives

$$\cosh(X/2) \int_0^{X/2} w(x) dx = \int_0^{X/2} w(x + X/2) dx \leq \int_{X/2}^X w(x) dy$$

and hence

$$(22) \quad \int_0^{X/2} w(x) dx \leq \frac{1}{\cosh(X/2)} \int_{X/2}^X w(x) dy.$$

Since p is decreasing

$$\int_0^X p(x) \cdot w(x) dx \leq p(0) \int_0^{X/2} w(x) dx + p(X/2) \int_{X/2}^X w(x) dx.$$

Combining this with (22) gives the claim. \square

Proposition 5.5. *Let $p : [0, X] \rightarrow \mathbb{R}^+$ be an increasing integrable function. Then*

$$\int_0^X w(x) dx \leq \frac{2}{p(X/2)} \int_0^X w(x) \cdot p(x) dx.$$

Proof. From (22) we have

$$\int_0^X w(x) dx \leq \left(\frac{1}{\cosh(X/2)} + 1 \right) \int_{X/2}^X w(x) dx.$$

Since p is increasing, we have

$$\int_{X/2}^X w(x) dx \leq \frac{1}{p(X/2)} \int_{X/2}^X w(x) \cdot p(x) dx.$$

By combining these inequalities and using the fact that $\cosh(X/2) \geq 1$, we obtain the claim. \square

6. LIMITS FOR ANALYTIC EIGENBRANCHES

In this section we prove that each real-analytic eigenvalue branch E_t converges. The proof consists of three main steps. First we prove that $t^2 \cdot E_t$ converges as t tends to zero. We use this to then prove that $t^{2k} E_t$ converges for some $k < 1$. Finally, we use this to prove that E_t converges. At each new stage, the previous estimate is used to control $h(t)$ along the eigenbranch.

6.1. Convergence relative to t^2 .

Proposition 6.1. *Let E_t be any eigenbranch. Then*

$$\lim_{t \rightarrow 0^+} t^2 \cdot E_t$$

exists and is finite.

Proof. Since $\cosh^2(x) \geq 1$, by comparing (11) and (15) we find that

$$t \cdot \dot{q}_U(\psi) \geq -q(\psi)$$

and by comparing (12) and (16)

$$t \cdot \dot{N}_U(\psi) \leq N(\psi).$$

Hence by Corollary 4.11 we have

$$(23) \quad \dot{E} \geq - \left(C + \frac{2}{t} \right) E$$

for some constant C . Lemma 4.8 then allows to conclude since in this case $F(t) = Ct^2 E_t$ for some constant C . \square

6.2. Convergence relative to t^{2k} . By Proposition 6.1, the parameter $h = t \cdot \sqrt{E}$ in the radial Mathieu equation associated to ψ is uniformly bounded in t . This will allow us to prove the following.

Theorem 6.2. *Let E_t be any eigenbranch. Then there exists $k < 1$ such that*

$$\lim_{t \rightarrow 0^+} t^{2k} \cdot E_t$$

exists and is finite.

Proof. It suffices to show that there exists $\kappa < 1$ so that

$$(24) \quad t \cdot \dot{N}_U(\psi) \leq \kappa \cdot N(\psi).$$

For then we could argue as in the proof of Proposition 6.1 where the ‘2’ that appears in (23) is replaced by $\kappa + 1$. In particular, the desired k equals $(\kappa + 1)/2$.

Lemma 6.3. *There exists $M > 0$ such that*

$$\int_{U_t} |\psi|^2 dx d\theta \leq M \int_{U_t} |\psi|^2 \sinh^2 x dx d\theta.$$

Assuming the lemma, we finish the proof of the theorem. Let $0 < \kappa < 1$ be such that $M = \kappa/(1 - \kappa)$. Then

$$(1 - \kappa) \int_{U_t} |\psi|^2 \sin^2 \theta dx d\theta \leq (1 - \kappa) \int_{U_t} |\psi|^2 dx d\theta \leq \kappa \int_{U_t} |\psi|^2 \sinh^2 x dx d\theta$$

and hence

$$\int_{U_t} |\psi|^2 \sin^2 \theta dx d\theta \leq \kappa \int_{U_t} |\psi|^2 \sinh^2 x dx d\theta + \kappa \int_{U_t} |\psi|^2 \sin^2 \theta dx d\theta.$$

Estimate (24) follows then by comparing (12) and (16). \square

Proof of Lemma 6.3. By Lemma 5.2, it suffices to prove that there exists M so that for all i

$$(25) \quad \int_0^{Y_t} |u_i|^2 dx \leq M \int_0^{Y_t} |u_i|^2 \sinh^2 x dx.$$

Let $p(x) = \sinh^2(x)$, and let $X = \sinh^{-1}(1)$. Since p is increasing on $[0, \infty)$, the infimum of $p(x)$ over $[X, \infty)$ equals $p(X) = 1$. In particular, if $t \leq r_0$, then $Y_t \geq X$, and we obtain

$$(26) \quad \int_X^{Y_t} |u_i|^2 p(x) dx \geq \int_X^{Y_t} |u_i|^2 dx.$$

By Proposition 6.1, $h = h(t)$ is uniformly bounded. Choose $i_0 \in \mathbb{Z}^+$ so that $i_0^2 \geq 2h^2(t) + 1/2$ for all sufficiently small t . Then since $b_i(t) \geq i^2$ —see Appendix A—and $\cosh^2(X) = 2$, we have that for all $x \in [0, X]$ and $i \geq i_0$

$$b_i(t) - h(t)^2 \cdot \cosh^2(x) \geq \frac{1}{2}.$$

Since ψ satisfies either Neumann or Dirichlet conditions along the slit, we have either $u'_i(0) = 0$ or $u_i(0) = 0$ for all i . Note also that p is increasing.

Thus, for $i \geq i_0$, we may apply Proposition 5.5 with $w = u_i^2$ to find that

$$\int_0^X |u_i|^2 dx \leq \frac{2}{p(X/2)} \int_0^X |u_i|^2 p(x) dx.$$

The lemma is thus proved for $i \geq i_0$ large enough.

For $i < i_0$, we note that by Proposition 6.1, h is bounded, and hence b_i is bounded. Thus, the claim for $i < i_0$ follows from the fact that the solution of the ordinary differential equation (18) with a fixed boundary condition depends continuously on parameters. \square

In the next section we make crucial use of the following variant of Theorem 6.2.

Corollary 6.4. *Let $h(t) = t \cdot \sqrt{E_t}$. There exists $\epsilon_0 > 0$ so that*

$$\lim_{t \rightarrow 0} t^{-\epsilon_0} h(t) = 0.$$

6.3. Convergence of E_t .

Theorem 6.5. *Let E_t be a real-analytic eigenbranch. Then*

$$\lim_{t \rightarrow 0^+} E_t$$

exists and is finite.

Proof. It suffices to show that there exists $\delta > 0$ so that for all sufficiently small t

$$(27) \quad \dot{q}_U(\psi) \geq -t^{\delta-1} \cdot q_U(\psi) - 2t^{\delta-1} \cdot E \cdot N_U(\psi),$$

and

$$(28) \quad \dot{N}_U(\psi) \leq t^{\delta-1} \cdot N_U(\psi).$$

For then by Corollary 4.11, there would exist $C > 0$ so that

$$(29) \quad \dot{E} \geq -(C + 4 \cdot t^{\delta-1}) E.$$

Since $t^{\delta-1}$ is integrable near $t = 0$, the claim would then follow from Lemma 4.8.

By comparing (11) to (13) and (12) to (16), we see that to prove (27) and (28) it suffices to prove that there exists $\delta > 0$ so that for sufficiently small t we have

$$(30) \quad \int_{U_t} \frac{|\partial_\theta \psi_t|^2}{\cosh^2 x} dx d\theta \leq t^\delta \left(\int_{U_t} |\partial_\theta \psi_t|^2 dx d\theta + 2t^2 \cdot E \cdot \int_{U_t} \psi^2 \sinh^2 x dx d\theta \right)$$

and

$$(31) \quad \int_{U_t} |\psi_t|^2 dx d\theta \leq t^\delta \int_{U_t} |\psi_t|^2 \sinh^2 x dx d\theta.$$

These estimates can be further reduced to estimates of Mathieu functions as in §5.2. In order to state these estimates, we first note that, by Corollary 6.4, there exists $\epsilon > 0$ such that

$$(32) \quad \lim_{t \rightarrow 0} t^{-2\epsilon} \cdot h(t) = 0.$$

We fix this ϵ in what follows.

Lemma 6.6. *Let u_i be a solution to the radial Mathieu equation (18) with $h = h(t)$ satisfying (32). There exists t_0 so that if $t < t_0$, then for all $i \geq 0$*

$$(33) \quad \int_0^{Y_t} |u_i|^2 dx \leq t^{\epsilon/2} \int_0^{Y_t} |u_i|^2 \sinh^2 x dx.$$

and for $i > 0$

$$(34) \quad \int_0^{Y_t} \frac{|u_i|^2}{\cosh^2 x} dx \leq t^{\epsilon/2} \int_0^{Y_t} |u_i|^2 dx$$

Assuming this lemma, we finish the proof of Theorem 6.5. Using (19) we see that (31) follows immediately from (33) with $\delta = \epsilon/2$.

To verify (30) we use (20) to find that

$$\int_{U_t} \frac{|\partial_\theta \psi|^2}{\cosh^2(x)} dx d\theta \leq \sum_{i=0}^{\infty} b_i \int_0^{Y_t} \frac{u_i^2(x)}{\cosh^2(x)} dx$$

In Appendix A we show that $b_0 \sim \frac{1}{2}h^2$ for small h . Hence since $h^2 = t^2 E$, we find that for small t

$$\begin{aligned} b_0 \int_0^{Y_t} \frac{u_0^2(x)}{\cosh^2(x)} dx &\leq h^2 \int_0^{Y_t} u_0^2(x) dx \\ &\leq E \cdot t^2 \cdot t^{\epsilon/2} \int_0^{Y_t} u_0^2(x) \sinh^2 x dx, \\ &\leq t^{\epsilon/2} \cdot t^2 \cdot E \int_{U_t} |\psi|^2 \sinh^2 x dx d\theta. \end{aligned}$$

Here we have used (33). Using (20) and (34), we find that

$$\begin{aligned} \sum_{i=1}^{\infty} b_i \int_0^{Y_t} \frac{u_i^2(x)}{\cosh^2(x)} dx &\leq t^\delta \sum_{i=1}^{\infty} b_i \int_0^{Y_t} u_i^2(x) dx \\ &\leq t^\delta \left(\int_{U_t} |\partial_\theta \psi|^2 dx d\theta + h^2 \int_{U_t} |\psi|^2 dx d\theta \right). \end{aligned}$$

Thus, since $h^2 = t^2 E$, and using (34) to estimate the last integral of the previous inequality, equation (30) follows with $\delta = \epsilon/2$. \square

6.4. The individual estimates. The remainder of this section is dedicated to proving Lemma 6.6. The behavior of the radial Mathieu function u_i for small h is qualitatively different if $i = 0$ than if $i \neq 0$. We prove the cases $i \neq 0$ with Lemma 6.7 and the case $i = 0$ with Lemma 6.9.

6.4.1. Estimates for $i \neq 0$.

Lemma 6.7. *There exists some t_0 and some constants C_1, C_2 such that for any $t < t_0$ and for any $i \neq 0$, we have*

$$(35) \quad \int_0^{Y_t} \frac{u_i^2(x)}{\cosh^2(x)} dx \leq C_1 \cdot t^\epsilon \int_0^{Y_t} u_i^2(x) dx,$$

and

$$(36) \quad \int_0^{Y_t} u_i^2(x) dx \leq C_2 \cdot t^\epsilon \int_0^{Y_t} u_i^2(x) \sinh^2(x) dx.$$

Proof. Define $X_t \in [0, \infty)$ implicitly by $\cosh(X_t) = t^{-\epsilon}$. Then we have

$$\int_{X_t}^{Y_t} \frac{u_i^2(x)}{\cosh^2(x)} dx \leq t^{2\epsilon} \int_{X_t}^{Y_t} u_i^2(x) dx.$$

Since $t^{2\epsilon} = o(t^\epsilon)$, It remains to estimate the integral over $[0, X_t]$.

By (32), we have that $h(t)^2 \cdot \cosh(X_t)^2$ tends to zero as t tends to zero. Since $i \neq 0$, we also have $b_i \geq 1$. (See Appendix A). Thus, for sufficiently small t ,

$$b_i(h(t)) - h(t)^2 \cdot \cosh^2(X_t) \geq \frac{1}{2}.$$

Therefore, we can apply the estimates of §5.2 with $w = u^2$ and $X = X_t$. By applying Proposition 5.4 with $p = \cosh^{-2}$, we find that

$$\int_0^{X_t} \frac{u_i^2(x)}{\cosh^2(x)} dx \leq \frac{2}{\cosh(X_t/2)} \int_0^{X_t} u_i^2(x) dx.$$

Since $\cosh(X_t/2) = \sqrt{(\cosh(X_t) + 1)/2} = \sqrt{(t^{-\epsilon} + 1)/2}$, we have

$$\int_0^{X_t} \frac{u_i^2(x)}{\cosh^2(x)} dx \leq \frac{2\sqrt{2} \cdot t^\epsilon}{1 + t^\epsilon} \int_0^{X_t} u_i^2(x) dx.$$

Estimate (35) then follows for $C_1 = 2\sqrt{2}$ and any t small enough so that $t^{2\epsilon} \leq C_1 \cdot t^\epsilon$.

The proof of the estimate (36) is similar. From the definition of X_t we have

$$\int_{X_t}^{Y_t} u_i^2 dx \leq \frac{2t^{2\epsilon}}{1 - t^{2\epsilon}} \int_{X_t}^{Y_t} u_i^2 \cdot \sinh(x)^2 dx,$$

and from Corollary 5.4 we find that

$$\int_0^{X_t} u_i^2(x) dx \leq \frac{8t^\epsilon}{1 - t^\epsilon} \int_0^{X_t} \sinh^2(x) \cdot u_i^2(x) dx,$$

the estimate follows with any $C_2 > 8$ and t small enough so that $\frac{2t^{2\epsilon}}{1 - t^{2\epsilon}}$ and $\frac{8}{1 - t^\epsilon}$ are less than C_2 . \square

Since equation (35) implies equation (34), and equation (36) implies equation (33), the estimates of Lemma 6.6 are proven for any index $i > 0$. It remains to prove (33) for $i = 0$.

6.4.2. *The estimate for $i = 0$.* Since $b_0(h)$ is real-analytic in h^2 , we can write

$$b_0(h) = a(h) \cdot h^2$$

for some analytic function a . It follows from Theorem 6.2 that $a(h(t))$ is bounded near $t = 0$. The Mathieu equation for $i = 0$ can thus be rewritten as

$$(37) \quad u_0''(x) = h(t)^2 \cdot (a(h(t)) - \cosh^2(x)) \cdot u_0(x).$$

We will treat as this equation as a perturbation of $u_0'' = 0$. To make this precise, we set $t_0 > 0$ and define

$$M(X) = \sup \{ |a(h(t)) - \cosh^2(x)| \mid 0 \leq x \leq X \text{ and } 0 \leq t \leq t_0 \}.$$

Lemma 6.8. *Given a solution u to (37) define R by*

$$R(x) = u(x) - (u(0) + u'(0) \cdot x).$$

Then for all $t \in (0, t_0)$, and for any positive weight p , the function R satisfies

$$\int_0^X |R(x)|^2 p(x) dx \leq h(t)^4 \cdot M(X)^2 \cdot X^4 \cdot L(X) \cdot \int_0^X |u(x)|^2 dx$$

where

$$L(X) = \sup_{x \in [0, X]} p(x).$$

Proof. By applying the method of variation of constants (or, equivalently, Duhamel's principle) we find that

$$R(x) = h(t)^2 \int_0^x (x-y) (a(h(t)) - \cosh^2(y)) u(y) dy.$$

Thus, for $t < t_0$ and $0 < x < X$

$$|R(x)|^2 \leq h^4 \cdot M(X)^2 \left(\int_0^x (x-y) \cdot |u(y)| dy \right)^2.$$

By applying the Bunyakovsky-Cauchy-Schwarz inequality we find that

$$\begin{aligned} |R(x)|^2 &\leq h^4 \cdot M^2(X) \left| \int_0^x (x-y)^2 dy \right| \cdot \left| \int_0^x u(y)^2 dy \right| \\ &\leq h^4 \cdot M^2(X) \cdot (|x|^3/3) \cdot \left| \int_0^x u(y)^2 dy \right| \end{aligned}$$

for all $|x| \leq X$. Thus,

$$\int_0^X |R(x)|^2 p(x) dx \leq h^4 \cdot M^2(X) \cdot X^3 \cdot L(X) \int_0^X \left| \int_0^x u(y)^2 dy \right| dx.$$

The desired estimate follows. \square

The following completes the proof of Lemma 6.6.

Lemma 6.9. *Let $\epsilon > 0$ be as in (32). There exists some constant C such that, if we let $u = u_0$ be a solution to the radial Mathieu equation (18) associated to $b_0(t)$ and $h(t)$, and if u satisfies either $u(0) = 0$ or $u'(0) = 0$, then for any sufficiently small t we have*

$$\int_0^{Y_t} u^2 \leq C \cdot t^\epsilon \int_0^{Y_t} u^2(x) \sinh^2(x) dx.$$

Proof. As above, let $X_t \geq 0$ be defined by $\cosh(X_t) = t^{-\epsilon}$. Let $a = u'(0)$ and $b = u(0)$. Since $u(x)^2 \leq 2(ax+b)^2 + 2R(x)^2$, by applying Lemma 6.8 with $p(x) \equiv 1$, we find that

$$(38) \quad \int_0^{X_t} u^2(x) dx \leq 2 \int_0^{X_t} (ax+b)^2 dx + K(t) \int_0^{X_t} u^2(x) dx$$

where

$$K(t) = 2 M^2(X_t) \cdot h(t)^4 \cdot X_t^4.$$

In other words,

$$(39) \quad \int_0^{X_t} u^2(x) dx \leq \frac{2}{1-K(t)} \int_0^{X_t} (ax+b)^2 dx,$$

provided that $K(t)$ is less than 1. As t tends to zero, $X_t \sim \epsilon |\ln t|$ tends to infinity, and hence $M(X_t) \sim \cosh^2(X_t) \sim t^{-2\epsilon}$. It follows that $K(t) \sim (t^{-\epsilon} \cdot h \cdot \epsilon |\ln h|)^4$. Thus, by (32) $K(t)$ tends to zero as t tends to zero. Therefore, for t sufficiently small

$$(40) \quad \int_0^{X_t} u^2(x) dx \leq 4 \int_0^{X_t} (ax+b)^2 dx.$$

If $u(0) = 0$, then $ax+b = u'(0) \cdot x$, and if $u'(0) = 0$, then $ax+b = u(0)$. A straightforward calculation gives a constant C such that for all sufficiently large X , we have

$$\int_0^X 1 dx \leq \frac{C \cdot X}{\sinh^2(X)} \int_0^X \sinh^2(x) dx.$$

and

$$\int_0^X x^2 dx \leq \frac{C \cdot X}{\sinh^2(X)} \int_0^X x^2 \sinh^2(x) dx.$$

Thus, since $\lim_{t \rightarrow 0} X_t = \infty$, for small t we have

$$(41) \quad \int_0^{X_t} (ax+b)^2 dx \leq \frac{C \cdot X_t}{\sinh^2(X_t)} \int_0^{X_t} (ax+b)^2 \sinh^2(x) dx.$$

By applying Lemma 6.8 with $p(x) = \sinh^2(x)$, we obtain

$$\int_0^{X_t} (ax+b)^2 \sinh^2(x) dx \leq 2 \int_0^{X_t} u(x)^2 \sinh^2(x) dx + \tilde{K}(t) \int_0^{X_t} u(x)^2 dx$$

where $\tilde{K}(t) = K(t) \cdot \sinh^2(X_t)$. By combining this with (40) and (41) we have

$$\int_0^{X_t} u^2(x) dx \leq \frac{8C \cdot X_t}{\sinh^2(X_t)} \int_0^{X_t} u^2(x) \sinh^2(x) dx + 4C \cdot X_t \cdot K(t) \int_0^{X_t} u^2(x) dx.$$

Arguing as above, one sees that $X_t \cdot K(t)$ tends to zero as t tends to zero. It follows that for all sufficiently small t , we have

$$\int_0^{X_t} u^2 dx \leq \frac{9C \cdot X_t}{\sinh^2(X_t)} \int_0^{X_t} u^2(x) \sinh^2(x) dx.$$

Note that $CX_t/\sinh^2(X_t) \leq t^\epsilon$ for sufficiently small t .

Since

$$\inf_{[X_t, Y_t]} \sinh^2(x) = \sinh^2(X_t) = t^{-2\epsilon} - 1,$$

we also have

$$\int_{X_t}^{Y_t} u^2(x) dx \leq \frac{t^{2\epsilon}}{1 - t^{2\epsilon}} \int_{X_t}^{Y_t} u^2(x) \sinh^2(x) dx,$$

The claim follows. \square

7. GENERIC SIMPLICITY FOR POLYGONS

In this section we combine the convergence of analytic eigenvalue branches with the convergence of eigenfunctions to generalize our earlier results [HlrJdg07] on spectral simplicity of simply-connected polygons to several settings.

7.1. Slits and simplicity. Our results on spectral simplicity for polygons depend on the following.

Proposition 7.1. *Let Σ_t be a slit of length $2t$ centered at a point p belonging to the interior of a Lipschitz domain $\Omega \subset \mathbb{R}^2$. Let D be a measurable subset of $\partial\Omega$ and let $D'_t \subset \Omega_\Sigma$ be either D or $D \cup \Sigma$. If the spectrum of q on $H_D^1(\Omega)$ is simple, then for all but countably many t , the spectrum of q on $H_{D'_t}^1(\Omega_{\Sigma_t})$ is simple.*

Proof. By Theorem 4.2, the eigenvalues vary analytically for $t > 0$. Hence it suffices to show that there does not exist a real-analytic eigenvalue branch E_t such that the dimension of the associated eigenspace V_t is greater than 1 for each $t > 0$.

Suppose that ψ_t and $\psi_t^* \in V_t$ denote normalized real-analytic eigenfunction branches that are mutually orthogonal for each t . By Theorem 6.5, the corresponding eigenvalue branch converges to some E_0 when t goes to 0. By Theorem 3.1, ψ_t and ψ_t^* converge to eigenfunctions ψ and ψ^* on Ω_Σ with the same eigenvalue E_0 . Since ψ_t and ψ_t^* are normalized and orthogonal for each t , the limits ψ and ψ^* are orthogonal. But this contradicts the assumption that the spectrum of Ω_Σ is simple. \square

Remark 7.2. The proof applies equally well if Ω is a slit domain with simple spectrum. Thus one can iterate the procedure, and prove that for all but countably choices of slit lengths, the domain Ω slit along finitely many slits has simple spectrum. Actually, if we take all the slits of the same length t , the spectrum is also simple for all but countably choices of t but to prove this result one has to adapt Theorem 6.5 to the case of several slits. This is easily done using the localization principle and the estimates we have proved.

By combining Theorem 7.1, Remark 7.2, and the main result of [HlrJdg07], we obtain the simplicity of the spectrum of the generic simply connected slit polygon. To be precise, let $\mathcal{S}_{n,k}$ be the set of simply connected n -gons P with k disjoint slits $\Sigma_1, \Sigma_2, \dots, \Sigma_k$. Note that the vertices of P and the endpoints of each slit $\Sigma_1, \Sigma_2, \dots, \Sigma_k$, determine the slit polygon $P_{\Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_k}$. Thus, $\mathcal{S}_{n,k}$ may be naturally identified with an open subset of $\mathbb{R}^{2n+2k+2k}$. In particular, $\mathcal{S}_{n,k}$ inherits a natural affine structure and Borel measure.

Theorem 7.3. *If $n \geq 4$ and $k \geq 0$, then almost every slit polygon in $\mathcal{S}_{n,k}$ has simple Dirichlet or Neumann spectrum.*

Proof. Let P be a simply connected n -gon. Let $C = \{c_1, \dots, c_k\}$ be a set of k distinct points in P , and let $L = \{\ell_1, \dots, \ell_k\}$ be a set of (not necessarily distinct) lines that pass through the origin in \mathbb{R}^2 . Let $A(P, C, L)$ be the set of slit polygons $P_{\Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_k}$ where Σ_i is centered at c_i and is parallel to ℓ_i . Note that a point in $A(P, C, L)$ is determined by the lengths of the slits.

The sets $A(P, C, L)$ provide a natural smooth foliation of $\mathcal{S}_{n,k}$ by k -dimensional planar sets. By Proposition 7.1 and Remark 7.2, if P has simple spectrum, then almost every slit polygon in $A(P, C, L)$ has simple spectrum. By the main result of [HlrJdg07], almost every simply connected n -gon has simple spectrum. The result follows by integrating transversely to this foliation using Fubini's theorem. \square

7.2. Polygons with mixed boundary conditions. Let P be a simply connected polygon, and let D be a union of a set of edges in the boundary, ∂P , of P . Recall from §2 that the eigenfunctions of the Dirichlet energy q on $H_D^1(P)$ satisfy Dirichlet

conditions on D and Neumann conditions on $\partial P \setminus D$. We say that (P, D) has simple spectrum if and only if the quadratic form q on $H_D^1(P)$ has simple spectrum.

Let v_1, v_2, \dots, v_n be a cyclic ordering of the edges of a simply connected polygon P . Let e_i denote the boundary edge joining the vertex v_i to v_{i+1} .⁴ Given a subset $\sigma \subset \{1, 2, \dots, n\}$, let $\mathcal{P}_{n,\sigma}$ denote the set of pairs (P, D) where P is a simply connected n -gon⁵ and

$$D = \bigcup_{i \in \sigma} e_i.$$

The set $\mathcal{P}_{n,\sigma}$ can be naturally identified with an open subset of \mathbb{R}^{2n} and hence has a natural Borel measure and affine structure.

Theorem 7.4. *If $n \geq 4$, then almost every polygon in $\mathcal{P}_{n,\sigma}$ has simple spectrum.*

Proof. By arguing as in [HlrJdg07], we see that it suffices to construct one polygon in $\mathcal{P}_{n,\sigma}$ that has simple spectrum.

We first construct such a polygon in the case of alternating boundary conditions, that is, we suppose that $i \in \sigma$ if and only if $i+1 \notin \sigma$ where as usual $n+1 \equiv 1$. Note that n is even in this case.

Let Q be the rectangle $[0, a] \times [0, 1]$, and let $D = \{0, 1\} \times [0, a]$. That is, we consider the eigenvalue problem on Q with Dirichlet conditions on the vertical edges and Neumann conditions on the horizontal edges. The set of functions of the form

$$\sin\left(\frac{m_1\pi}{a} \cdot x\right) \cdot \cos(m_2\pi \cdot y),$$

where $m_1 > 0$ and $m_2 \geq 0$ are integers, is an orthonormal basis of eigenfunctions. Thus, the set of eigenvalues is $\{(m_1\pi/a)^2 + (m_2\pi)^2\}$. In particular, (Q, D) has simple spectrum if and only if $a^2 \notin \mathbb{Q}$.

Let c_1, c_2, \dots, c_k be $k = n/2 - 2$ distinct points on the horizontal segment $[0, a] \times \{1/2\}$. By Theorem 7.1, there exist horizontal slits $\Sigma_1, \dots, \Sigma_k$ centered respectively at c_i such that $(Q_{\Sigma_1 \cup \dots \cup \Sigma_k}, D')$ has simple spectrum where $D' = D \cup \Sigma_1 \cup \dots \cup \Sigma_k$. See Figure 1.

The slit domain $Q_{\Sigma_1 \cup \dots \cup \Sigma_k}$ has a reflection symmetry τ induced by $(x, y) \mapsto (x, 1-y)$. A standard argument shows that restricting functions on $Q_{\Sigma_1 \cup \dots \cup \Sigma_k}$ to the rectangle $Q' = [0, a] \times [0, 1/2]$ defines a bijection between τ^* -invariant eigenfunctions of q on $H^1(Q_{\Sigma_1 \cup \dots \cup \Sigma_k}, D')$ and the eigenfunctions of q on $H_{D'}^1(Q')$.

Therefore, since the spectrum of $(Q_{\Sigma_1 \cup \dots \cup \Sigma_k}, D')$ is simple, the spectrum of (Q', D') is also simple. By placing vertices at the endpoints of the slit, we may regard the rectangle Q' as a polygon with $n = 2k + 4$ vertices. From this viewpoint, the boundary conditions given by D' alternate as desired.

Given a general subset $\sigma \subset \{1, 2, \dots, n\}$, let F be the finite set obtained by identifying i and $i+1$ if they either both belong to σ or if they both do not belong to σ . The cyclic ordering of $\{1, 2, \dots, n\}$ induces a cyclic ordering on F , and hence we may identify F with $\{1, 2, \dots, n'\}$ for some $n' \leq n$. Note that n' is either 1 or even. Let $\sigma' \subset \{1, 2, \dots, n'\}$ denote the set obtained by identifying $i \in \sigma$ with $i+1$ if it also belongs to σ .

The set $\sigma' \subset \{1, 2, \dots, n'\}$ is alternating, and hence, if $n' \geq 4$, there exists an element (P', D) in $\mathcal{P}_{n',\sigma'}$ with simple spectrum. By judiciously adding vertices to

⁴Here e_n joins v_n and v_1 .

⁵By *simply connected polygon*, we mean a compact set whose boundary consists of finitely many line segments and whose interior is simply connected.

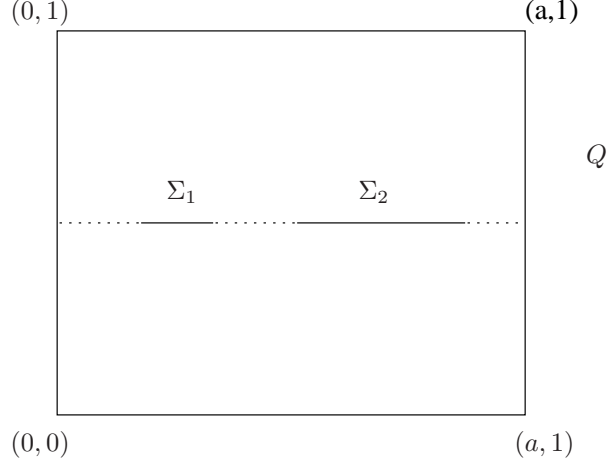


FIGURE 1. The slit rectangle

the boundary edges of the n' -gon P , we obtain an n -gon P' such that the set D corresponds to σ . Thus (P, D) is an element of $\mathcal{P}_{n,\sigma}$ with simple spectrum.

If $n' = 2$, then we consider the eigenvalue problem on (Q, D) where $Q = [0, a] \times [0, 1]$ and D is either $\{0\} \times [0, 1]$ or its complement. In either case a basis of eigenfunctions can be constructed by taking products of sines and cosines. By making an appropriate choice of a , one finds that the spectrum of (Q, D) is simple. By adding vertices appropriately to the boundary edges of Q , we obtain $(Q', D) \in \mathcal{P}_{n,\sigma}$ with simple spectrum.

The case $n' = 1$ is the main result of [HlrJdg07] and follows from the same construction. \square

7.3. Mutliply connected polygons. We first make precise the definition of multiply connected polygon. Let P_0, P_1, \dots, P_k be a finite collection of simply connected polygons. Assume that

- (1) P_i is contained in the interior of P_0 for all $i > 0$, and
- (2) $P_i \cap P_j = \emptyset$ for all $i, j \geq 0$, $i \neq j$,

The *multiply connected polygon* P determined by P_0, P_1, \dots, P_k is obtained by removing the interiors of the polygons P_1, \dots, P_k from the polygon P_0 . In other words, P is the closure of

$$P_0 \setminus \left(\bigcup_{i=1}^k P_i \right).$$

Let $\vec{n} = (n_0, n_1, \dots, n_k)$ denote a vector of integers with $n_i \geq 3$ for each i . Let $\mathcal{P}(\vec{n})$ denote the set of all collections of polygons P_0, P_1, \dots, P_k satisfying (1) and (2) above and such that for each i , the polygon P_i has n_i (ordered) vertices $v_{i,1}, \dots, v_{i,n_i}$. Since the (ordered) vertices determine the polygon, $\mathcal{P}(\vec{n})$ is naturally in bijective correspondence with an open subset of \mathbb{R}^d where $d = 2n_0 + \dots + 2n_k$. In particular, $\mathcal{P}(\vec{n})$ inherits an affine structure and a Borel measure.

Proposition 7.5. *The space $\mathcal{P}(\vec{n})$ is path connected.*

Proof. In [HlrJdg07], we proved that the space \mathcal{P}_n of simply connected n -gons is connected using a construction which we will call ‘deleting a vertex’. In particular, given a simply connected n -gon P , we constructed a linear path $t \mapsto P(t)$ of n -gons in \mathcal{P}_n with $P(0) = P$ and such that $P(1)$ has three consecutive vertices that belong to the same boundary edge. See Figure 2. We can regard the polygon $P(1)$ as an element of \mathcal{P}_{n-1} . Thus, since the space of triangles, \mathcal{P}_3 , is connected, \mathcal{P}_n is connected for $n \geq 3$ by induction.

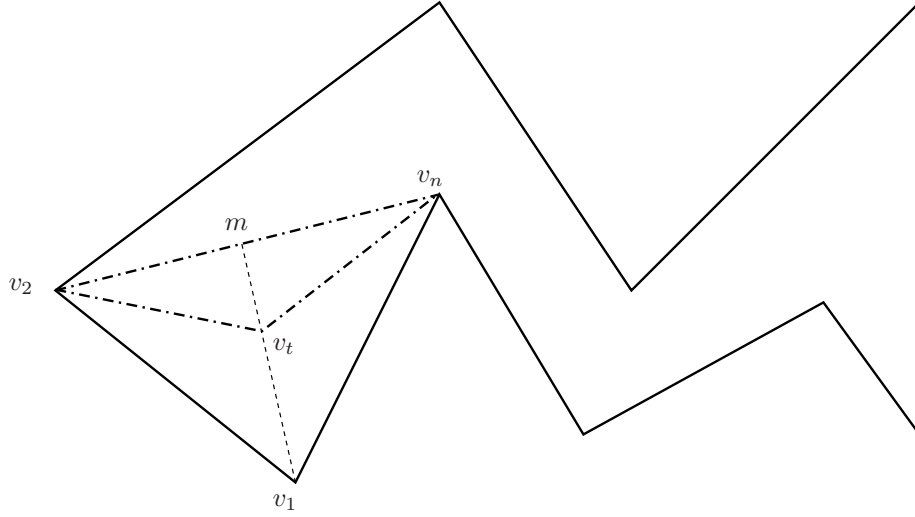


FIGURE 2. Deleting a vertex

Suppose now that $P \in \mathcal{P}(\vec{n})$ is defined by the simply connected polygons P_0, P_1, \dots, P_k where $n_i > 3$ for some $i > 0$. Let $t \mapsto P_i(t)$ denote the path from $P_i = P_i(0)$ to a polygon $P_i(1)$ having three consecutive vertices on the same edge. Let $t \mapsto P(t)$ denote the path of polygons determined by $P_1, \dots, P_{i-1}, P_i(t), P_{i+1}, \dots, P_k$. The polygon $P(1)$ may be regarded as an element of $\mathcal{P}(n_0, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_k)$.

Therefore, by induction on the vector \vec{n} , we find that it suffices to prove that the space $\mathcal{P}(n, 3, 3, 3, \dots, 3)$ is connected. By rescaling the ‘interior’ triangles if necessary, we may delete vertices of the polygon P_0 to obtain a path to an element of $\mathcal{P}(3, 3, 3, 3, \dots, 3)$. An elementary argument then gives that $\mathcal{P}(3, 3, 3, 3, \dots, 3)$ is connected. \square

Theorem 7.6. *If $n_0 \geq 4$, then almost every polygon in $\mathcal{P}(\vec{n})$ has simple spectrum.*

Proof. Since $\mathcal{P}(\vec{n})$ is connected, by arguing as in [HlrJdg07], we see that it suffices to construct one polygon in $\mathcal{P}(\vec{n})$ that has simple spectrum.

Let P belong to $\mathcal{P}(4, 3, 3, 3, \dots, 3)$. By judiciously adding vertices to the boundary edges of P , we may regard P as an element of $\mathcal{P}(n_0, n_1, n_2, n_3, \dots, n_k)$ where $n_0 \geq 4$ and $n_i \geq 3$ for $i > 0$. Thus, it will suffice to prove that there exists some $P \in \mathcal{P}(4, 3, 3, 3, \dots, 3)$ such that P has simple spectrum.

By Theorem 7.3 there exists a quadrilateral Q with k slits $\Sigma_1, \Sigma_2, \dots, \Sigma_k$ so that $Q_{\Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_k}$ has simple spectrum. For each i , choose a point p_i so that

the convex hulls, P_i , of $\{p_i\} \cup \Sigma_i$ satisfy conditions (1) and (2) above. Let m_i be the midpoint of Σ_i and define the path $x_i(t) = tp_i + (1-t)m_i$. Define $P_i(t)$ to be the convex hull of $\{x_i\} \cup \Sigma_i$. Let $P(t)$ be the multiply connected polygon defined by $Q, P_1(t), P_2(t), \dots, P_k(t)$. See Figure 3. Note that $P(t)$ is an element of $\mathcal{P}(4, 3, 3, \dots, 3)$ for $t > 0$ and that $P(0)$ corresponds to $(Q, \Sigma_1, \Sigma_2, \dots, \Sigma_n)$.

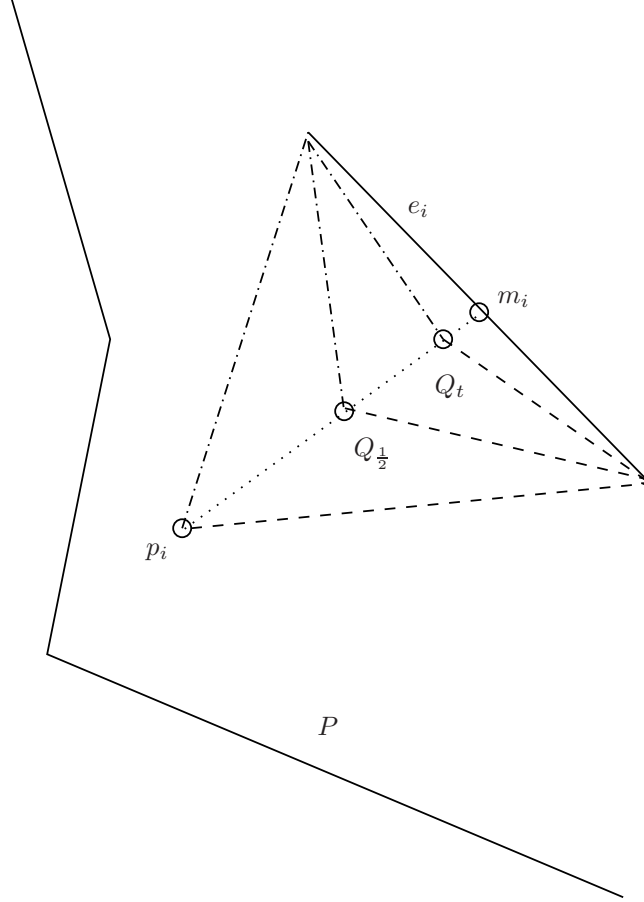


FIGURE 3. Perturbing a slit into a triangle

We define a linear path $t \mapsto f_t$ of piecewise linear homeomorphisms mapping $P(1/2)$ onto $P(t)$ for each $t \in [0, 1/2]$. In particular, let $Q_i(t)$ be the quadrilateral $P_i(1) \setminus P_i(t)$, and let f_t^i be the ‘obvious’ piecewise linear homeomorphism that maps $Q_i(1/2)$ onto $Q_i(t)$. Define $f_t : P(1/2) \rightarrow P(t)$ by

$$f_t(x) = \begin{cases} x, & \text{if } x \notin \bigcup_i Q_i(1/2) \\ f_t^i(x), & \text{if } x \in Q_i(1/2). \end{cases}$$

By Lemma 2.1 in [HlrJdg07], the eigenvalues of $P(t)$ vary real-analytically in t . Since the spectrum of $P(0)$ is simple, there exists $t_0 > 0$ such that the spectrum of $P(t_0)$ is simple. The claim is proven. \square

APPENDIX A. MATHIEU FUNCTIONS

For the convenience of the reader we provide basic facts about Mathieu functions that are used in the present paper. For additional information, we refer the reader to [Mth68] and [MrsFsh]. Our approach is based on analytic perturbation theory. See Example VII.3.4 in [Kato].

Suppose that $\Delta\psi = E \cdot \psi$. A straightforward computation shows that

$$(42) \quad -(\partial_z^2 + \partial_\theta^2)(\psi \circ F_t) = t^2 \cdot E \cdot (\cosh(z)^2 - \cos^2(\theta)) \cdot (\psi \circ F_t).$$

Let $h = t\sqrt{E}$. The method of separation of variables leads one to consider the operator

$$A_h = -\frac{d^2}{d\theta^2} + h^2 \cos^2(\theta)$$

acting self-adjointly on $L^2(\mathbb{R}/(2\pi\mathbb{Z}), d\theta)$.

Proposition A.1 (Angular Mathieu functions). *For each $i \in \mathbb{Z}$, there exist unique real-analytic paths $v_i : \mathbb{R} \rightarrow L^2(\mathbb{R}/(2\pi\mathbb{Z}), d\theta)$ and $b_i : \mathbb{R} \rightarrow \mathbb{R}$ so that for each $h \in \mathbb{R}$*

(a) *$v_i(h)$ is an eigenfunction for A_h with eigenvalue $b_i(h)$:*

$$A_h(v_i(h)) = b_i(h) \cdot v_i(h).$$

(b) *$v_i(h)$ has unit norm in $L^2(\mathbb{R}/(2\pi\mathbb{Z}), d\theta)$,*

(c) *the span of $\{v_i(h) \mid i = 0, 1, 2, \dots\}$ is dense in $L^2(\mathbb{R}/(2\pi\mathbb{Z}), d\theta)$,*

(d) *For i odd*

$$v_i(h)(\theta) = \pi^{-\frac{1}{2}} \cdot \sin(i\theta) + O(h^2),$$

and for i even

$$v_i(h)(\theta) = \pi^{-\frac{1}{2}} \cdot \cos(i\theta) + O(h^2).$$

(e) *For i odd*

$$b_i(h) = i^2 + 2h^2 + O(h^4),$$

and for i even

$$b_i(h) = i^2 + \frac{1}{2}h^2 + O(h^4)$$

(f) *For any i and any h , we have $b_i(h) \geq b_i(0)$.*

Remark A.2. The functions $v_i(h)$ are known classically as angular Mathieu functions. and we will use indifferently the notations $v_i(h)$ and $v_{i,h}$. See for example [MrsFsh].

Proof. The path $h \mapsto A_h$ is an analytic family of compactly resolved operators. Hence the existence of paths v_i and b_i satisfying (a), (b), and (c) follows from Theorem VII.3.9 on page 392 in [Kato].

The k^2 -eigenspace of $A_0 = \partial_\theta^2$ is spanned by $\mathcal{B}_i = \{\sin(i\theta), \cos(i\theta)\}$. Let P_i be the orthogonal projection onto this eigenspace. Let \ddot{A} denote the second derivative of $h \mapsto A_h$ evaluated at $h = 0$. One computes the matrix of $P_i \ddot{A} P_i$ with respect to the orthonormal basis \mathcal{B}_i to be

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Uniqueness of the paths as well as property (d) then follow from analytic perturbation theory [Kato]. The derivative of b_i is given by

$$\frac{d}{dh}b_i(h) = 2h \int_0^{2\pi} \cos^2(\theta) v_{i,h}(\theta)^2 d\theta.$$

Thus b_i is increasing for positive h and decreasing for negative h showing that $h = 0$ is a global minimum. \square

Corollary A.3. *Let ψ be an eigenvector of Δ with Dirichlet boundary condition on the slit (resp. Neumann) then*

$$\psi(z, \theta) = \sum_i u_{i,h}(z) \cdot v_{i,h}(\theta).$$

where

$$v_{i,h} = v_i(h)$$

are as above, and $u = u_{i,h}$ satisfies the ordinary differential equation

$$(43) \quad -u''(z) + (b_i(h) - h^2 \cosh^2(z)) \cdot u(z) = 0,$$

with Dirichlet boundary condition at $z = 0$ (resp. Neumann).

Proof. Each eigenfunction ψ is smooth and hence by Proposition A.1

$$\psi(z, \theta) = \sum_i u_{i,h}(z) \cdot v_{i,h}(\theta).$$

where $h^2 = t^2 E$, and

$$u_{i,h}(r) = \int_0^{2\pi} \psi(z, \theta) \cdot v_{i,h}(\theta) d\theta.$$

One obtains (18) by using (42) and integrating by parts. \square

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